Problem 8-1: Scales of western-boundary current

From expression (8-20) giving the meridional velocity,

\[ v = \frac{1}{\rho_0 \beta y} \frac{d\tau}{dy} \left[ -1 + \frac{2 \beta y L_1}{f_0 d} \exp\left( -\frac{2 \beta y}{f_0 d} \right) \right]. \]

we note that the exponential decay of the current from its maximum value at the coast to its Sverdrup value in the interior occurs over the e-folding scale (inverse of the coefficient of z in the exponential function).

\[ \frac{f_0 d}{2 \beta y} \]

This length can be considered as the width of the boundary current. Note that it does not depend on \( L_1 \), the width of the basin. The velocity scale is obtained by examining the value of \( v \) at the coast \((x=0)\). Neglecting the small Sverdrup-flow contribution, we have:

\[ v_{\text{coast}} = \frac{1}{\rho_0 \beta y} \frac{d\tau}{dy} \frac{2 \beta y L_1}{f_0 d} = \frac{2L_2}{\rho_0 f_0 d} \frac{d\tau}{dy} \]

Estimating the wind-stress curl \( d\tau/dy \) at \( \tau_0/2L_2 \) (the factor 2 is optional), we derive the following velocity scale

\[ V = \frac{L_2 \tau_0}{\rho_0 f_0 d L_2} \]

Note that it is inversely proportional to the small Ekman depth \( d \) (and is therefore large) and is proportional to the basin width \( L_1 \). This last conclusion could have been anticipated: The wider the basin, the greater the southward Sverdrup flow that must be returned northward by the boundary current.
Show that for the gravity wave modes supported in the shallow water system linearized about a basic state at rest, the complex amplitudes of the zonal and meridional winds are given by:

\[ \hat{u} = \frac{g H}{\omega^2 - f^2}(\omega k - i l) \quad \text{and} \quad \hat{v} = \frac{g H}{\omega^2 - f^2}(\omega l - if k), \]

respectively.

First, start with the linearized equations:

\[ \frac{\partial \hat{u}'}{\partial t} - f \hat{v}' = -g \frac{\partial \hat{h}'}{\partial x}, \quad \frac{\partial \hat{v}'}{\partial t} + f \hat{u}' = -g \frac{\partial \hat{h}'}{\partial y}, \quad \frac{\partial \hat{h}'}{\partial t} + H(\frac{\partial \hat{u}'}{\partial x} + \frac{\partial \hat{v}'}{\partial y}) = 0, \]

where

\[ \hat{u}' = \text{Re}[\hat{u}e^{i(\omega k + ly - \omega t)}], \quad \hat{v}' = \text{Re}[\hat{v}e^{i(\omega k + ly - \omega t)}], \quad \text{and} \quad \hat{h}' = \text{Re}[\hat{h}e^{i(\omega k + ly - \omega t)}]. \]

After taking the derivatives, the linearized equations can be written as listed below.

\[
\begin{align*}
-\hat{u}o\omega e^{i(\omega k + ly - \omega t)} - f\hat{v}e^{i(\omega k + ly - \omega t)} &= -g\hat{h}ke^{i(\omega k + ly - \omega t)} \\
-\hat{v}o\omega e^{i(\omega k + ly - \omega t)} + f\hat{u}e^{i(\omega k + ly - \omega t)} &= -g\hat{h}le^{i(\omega k + ly - \omega t)} \\
-\hat{h}o\omega e^{i(\omega k + ly - \omega t)} + He^{i(\omega k + ly - \omega t)}(\hat{u} + i\hat{v}) &= 0
\end{align*}
\]

Simplifying these equations we get:

A. \(-\hat{u}o\omega - f\hat{v} = -g\hat{h}k\)
B. \(-\hat{v}o\omega + f\hat{u} = -g\hat{h}l\)
C. \(-\hat{h}o\omega + H(\hat{u} + i\hat{v}) = 0\)

In order to find \(\hat{u}\) let's take \(io\hat{A} - f\hat{B}\), which yields

\[
-\hat{u}^2\omega^2 - i\omega f\hat{v} + i\omega f\hat{v} - f^2\hat{u} = -g\hat{h}^2k\omega + g\hat{h}il \Rightarrow \hat{u}(\omega^2 - f^2) = g\hat{h}(\omega k + il)
\]

\[ \Rightarrow \hat{u} = \frac{g \hat{h}}{(\omega^2 - f^2)}(\omega k + il). \]

To find \(\hat{v}\), we can take \(f\hat{A} + io\hat{B}\), which yields

\[
-\hat{u}f\omega - f^2\hat{v} + i\omega f\hat{v} + i\omega f\hat{u} = -g\hat{h}f\omega - g\hat{h}o\omega^2l \Rightarrow \hat{v}(\omega^2 - f^2) = g\hat{h}(\omega l - if k)
\]

\[ \Rightarrow \hat{v} = \frac{g \hat{h}}{(\omega^2 - f^2)}(\omega l - ifk). \]
29) Using \( \hat{u} = \frac{gh}{\omega^2 - f^2} (\omega k + ifl) \) and \( \hat{v} = \frac{gh}{\omega^2 - f^2} (\omega l - ifk) \), show that

\[
u' = \frac{gh}{\omega^2 - f^2} [\omega k \cos(kx + ly - \omega t) - fl \sin(kx + ly - \omega t)], \quad \text{and}
\]
\[
\nu' = \frac{gh}{\omega^2 - f^2} [\omega l \cos(kx + ly - \omega t) + kf \sin(kx + ly - \omega t)].
\]

* Solve for \( \nu' \):

\[
u' = \text{Re}[\hat{u} e^{(kx + ly - \omega t)}] = \text{Re}[\hat{u} (\cos(kx + ly - \omega t) + i \sin(kx + ly - \omega t))]
\]

Substitute \( \hat{u} \) into the equation to get:

\[
u' = \text{Re}[\frac{gh}{w^2 - f^2} ((\omega k + ifl) (\cos(kx + ly - \omega t) + i \sin(kx + ly - \omega t))].
\]

Expand to get

\[
\text{Re}[\frac{gh}{w^2 - f^2} (\omega k \cos(kx + ly - \omega t) + i \omega k \sin(kx + ly - \omega t) + ifl \cos(kx + ly - \omega t) + i^2 fl \sin(kx + ly - \omega t))]
\]

Taking the real part of the above expression yields:

\[
u' = \frac{gh}{w^2 - f^2} (\omega k \cos(kx + ly - \omega t) - fl \sin(kx + ly - \omega t)).
\]

* Solve for \( \nu' \):

\[
u' = \text{Re}[\hat{v} e^{(kx + ly - \omega t)}] = \text{Re}[\hat{v} (\cos(kx + ly - \omega t) + i \sin(kx + ly - \omega t))]
\]

Substitute \( \hat{v} \) into the equation and expand to get:

\[
\text{Re}[\frac{gh}{w^2 - f^2} (\omega l \cos(kx + ly - \omega t) + i \omega l \sin(kx + ly - \omega t) - ifk \cos(kx + ly - \omega t) - i^2 fk \sin(kx + ly - \omega t))]
\]

Finally, taking the real part yields:

\[
v' = \frac{gh}{w^2 - f^2} (\omega l \cos(kx + ly - \omega t) + fk \sin(kx + ly - \omega t)).
\]
31) Linearize the 2D, non-divergent, barotropic, vorticity equation on a β-plane and find the dispersion relationship for \( \overline{\psi} = -Uy \) and \( \psi' = \text{Re}[\hat{\psi}e^{i(kx+ly-\omega t)}] \).

First, linearize the vorticity equation: \( \frac{\partial \zeta}{\partial t} = -u \frac{\partial \zeta}{\partial x} - v \frac{\partial \zeta}{\partial y} - \beta v' \).

Set \( \zeta = \overline{\zeta} + \zeta' \), \( u = \overline{u} + u' \), and \( v = \overline{v} + v' \).

Set the basic state as \( \frac{\partial \overline{\zeta}}{\partial t} = -\overline{u} \frac{\partial \overline{\zeta}}{\partial x} - \overline{v} \frac{\partial \overline{\zeta}}{\partial y} - \beta \overline{v} = 0 \).

The linearized equation becomes:

\[
\frac{\partial}{\partial t} (\overline{\zeta} + \zeta') = -(\overline{u} + u') \frac{\partial}{\partial x} (\overline{\zeta} + \zeta') - (\overline{v} + v') \frac{\partial}{\partial y} (\overline{\zeta} + \zeta') - \beta (\overline{v} + v').
\]

Expanding this, we now have:

\[
\frac{\partial \overline{\zeta}}{\partial t} + \frac{\partial \zeta'}{\partial t} = -(\overline{u} \frac{\partial \overline{\zeta}}{\partial x} + \overline{u} \frac{\partial \zeta'}{\partial x} + u' \frac{\partial \zeta'}{\partial x} + u \frac{\partial \overline{\zeta}}{\partial x} + u' \frac{\partial \overline{\zeta}}{\partial x}) - (\overline{v} \frac{\partial \overline{\zeta}}{\partial y} + \overline{v} \frac{\partial \zeta'}{\partial y} + v' \frac{\partial \zeta'}{\partial y} + v \frac{\partial \overline{\zeta}}{\partial y} + v' \frac{\partial \overline{\zeta}}{\partial y}) - \beta \overline{v} - \beta v'.
\]

Now let’s simplify. Start by removing the basic state since it has been set to zero.

\[
\frac{\partial \zeta'}{\partial t} = -(\overline{u} \frac{\partial \zeta'}{\partial x} + u' \frac{\partial \zeta'}{\partial x} + u \frac{\partial \zeta'}{\partial x} - (\overline{v} \frac{\partial \zeta'}{\partial y} + v' \frac{\partial \zeta'}{\partial y} + v \frac{\partial \zeta'}{\partial y}) - \beta v'.
\]

Now get rid of the non-linear terms to get the final equation.

\[
\frac{\partial \zeta'}{\partial t} = -\overline{u} \frac{\partial \zeta'}{\partial x} - u' \frac{\partial \zeta'}{\partial x} - \overline{v} \frac{\partial \zeta'}{\partial y} - v' \frac{\partial \zeta'}{\partial y} - \beta v'.
\]

In order to find the dispersion relationship, let us re-write this equation in terms of \( \psi \). Recall: \( \zeta = \nabla^2 \psi \), \( u = -\frac{\partial \psi}{\partial y} \), and \( v = \frac{\partial \psi}{\partial x} \). Substituting these in we get,

\[
\frac{\partial}{\partial t} (\nabla^2 \psi') = \frac{\partial \overline{\psi}}{\partial y} \frac{\partial \psi}{\partial x} (\nabla^2 \psi') + \frac{\partial \psi'}{\partial y} \frac{\partial \psi}{\partial x} (\nabla^2 \psi) - \frac{\partial \overline{\psi}}{\partial x} \frac{\partial \psi}{\partial y} (\nabla^2 \psi') - \frac{\partial \psi'}{\partial x} \frac{\partial \psi}{\partial y} (\nabla^2 \psi) - \beta \frac{\partial \psi'}{\partial x}.
\]

However, we know that \( \overline{\psi} = -Uy \), so \( \frac{\partial \overline{\psi}}{\partial x} = \nabla^2 \overline{\psi} = 0 \) and \( \frac{\partial \overline{\psi}}{\partial y} = -U \), giving us:

\[
\frac{\partial}{\partial t} (\nabla^2 \psi') = -U \frac{\partial}{\partial x} (\nabla^2 \psi') - \beta \frac{\partial \psi'}{\partial x}.
\]
Now use $\psi' = \text{Re}[\phi e^{i(kx + ly - \omega t)}]$ and take the appropriate derivatives to get:

$$-i\omega (i^2 k^2 + i^2 l^2) \hat{\phi} e^{i(kx + ly - \omega t)} = -Uik(i^2 k^2 + i^2 l^2) \hat{\phi} e^{i(kx + ly - \omega t)} - \beta ik \hat{\phi} e^{i(kx + ly - \omega t)}$$

This simplifies to

$$\omega (k^2 + l^2) = Uk(k^2 + l^2) - \beta k.$$

To get the dispersion relationship, solve for $\omega$.

$$\omega = \frac{Uk(k^2 + l^2) - \beta k}{(k^2 + l^2)},$$

which can finally be simplified to

$$\omega = Uk - \frac{\beta k}{(k^2 + l^2)}.$$

The phase speed of these waves, $c = \frac{\omega}{k} = U - \frac{\beta}{(k^2 + l^2)}$, depends on the wave number; therefore, these waves are dispersive.

32) (Holton 7.1)

Show that $F(x) = \text{Re}[C \exp(aimx)]$ can also be written as $F(x) = |C| \cos(m(x - x_o))$,

where $x_o = m^{-1} \sin^{-1}(\frac{C_i}{|C|})$.

First, we know that $F(x) = \text{Re}[C \exp(aimx)]$ can be expanded to

$F(x) = \text{Re}[C \cos(mx) + iC \sin(mx)]$, where $C = C_r + iC_i$.

$\Rightarrow F(x) = \text{Re}[C_r \cos(mx) + iC_i \cos(mx) + iC_r \sin(mx) - C_i \sin(mx)]$

Taking the real part of the equation we get,

$F(x) = C_r \cos(mx) - C_i \sin(mx)$

Now we need to find $C_r$ and $C_i$.

From $x_o = m^{-1} \sin^{-1}(\frac{C_i}{|C|})$ we get $C_i = |C| \sin(mx_o)$.

Define $C_r = |C| \cos(mx_o)$ and plug them into $F(x)$. The resulting equation is:

$F(x) = \text{Re}[|C| \cos(mx) \cos(mx_o) - |C| \sin(mx) \sin(mx_o)]$
Using the trig identity, \( \cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a - b) \), we get the final equation:

\[
F(x) = |C|\cos(m(x - x_0)).
\]

33) (Holton 7.2)

Show that \( \psi = A\cos(kx - vt - kx_o)\exp(\alpha t) \) can be written as \( \psi = \text{Re}[Be^{ik(x - ct)}] \).

Find \( B_r, B_i, c_r, c_i \) in terms of \( A, \alpha, v, \) and \( x_o \).

By defining \( C = c_r + ic_i \) we can re-write \( \psi \) as,

\[
\psi = \text{Re}[Be^{ik(x - ct)}]e^{kc_i t}.
\]

Using the rule from problem 7.1 we can further write,

\[
\psi = \text{Re}[Be^{ik(x - ct)}]e^{kc_i t} = \text{Re}[|B|\cos(k(x - c_r t) - kx_o)]e^{kc_i t}
\]

Now, let \( v = kc_r, \) and \( \alpha = kc_i. \)

\[
=> \psi = \text{Re}[|B|\cos(kx - vt - kx_o)]\exp(\alpha t)
\]

Similarly to 7.1, define \( x_o = -k^{-1}\sin^{-1}\left(\frac{B_i}{|B|}\right) \).

Giving us \( B_r = A\cos(kx_o) \) and \( B_i = A\sin(kx_o) \).

Now we have,

\[
\psi = A\cos(kx - vt - kx_o)\exp(\alpha t).
\]
Problem 6-2: Kelvin wave in the Yellow Sea

The Kelvin-wave speed is

\[ c = \sqrt{gH} = \sqrt{(9.81 \text{ m/s}^2)(50 \text{ m})} = 22.15 \text{ m/s}, \]

and the time to travel the distance \( L = 2600 \text{ km} = 2.6 \times 10^6 \text{ m} \) is

\[ T = \frac{L}{c} = \frac{2.6 \times 10^6 \text{ m}}{22.15 \text{ m/s}} = 1.174 \times 10^5 \text{ s} = 32\frac{1}{2} \text{ hours}. \]

Problem 6-6: Equatorial wave

If there is no meridional velocity \( (v=0) \), the equations reduce to

\[ \frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x}, \quad \frac{\partial \eta}{\partial x} = -g \frac{\partial \eta}{\partial y} \quad \text{and} \quad \frac{\partial \eta}{\partial t} + Hu = 0. \]

The first and last equations are identical to those in the Kelvin-wave theory, except for a permutation of coordinates, and admit the solutions

\[ u = U_1(x - ct, y) + U_2(x - ct, y), \]

\[ \eta = -\sqrt{\frac{H}{g}} U_1(x - ct, y) + \sqrt{\frac{H}{g}} U_2(x - ct, y), \]

where \( c = \sqrt{gH} \) is the zonal wave speed. Substitution of this solution in the remaining equation yields

\[ \frac{\partial U_1}{\partial y} = \frac{\partial U_2}{\partial y} = \frac{-g}{c} \eta U_1. \]

Solving for the meridional structure of these waves, we find that the \( U_1 \) function grows exponentially away from the equator and must be rejected. By contrast, the \( U_2 \) function decays away from the equator according to

\[ U_2 = U \exp \left( -\frac{\rho g^2}{2c} \right), \]

where \( U \) is an arbitrary function of \( (x-ct) \). The final solution is:

\[ u = U(x - ct) \exp \left( -\frac{\rho g^2}{2c} \right), \]

\[ v = 0, \]

\[ \eta = \sqrt{\frac{H}{g}} U(x - ct) \exp \left( -\frac{\rho g^2}{2c} \right). \]

This solution corresponds to an eastward propagating wave with maximum amplitude along the equator and decaying symmetrically away from it in both hemispheres. Because of its decay away from the equator, the wave can be said to be trapped; the trapping distance is provided by the values of \( \rho \) for which the exponent reaches unity, which is

\[ \text{trapping distance} = \frac{\sqrt{2c}}{\rho g}. \]

The wave is non-dispersive and propagates at the speed \( c = \sqrt{gH} \) equal to that of a surface gravity wave.

In view of the above properties, this wave is obviously the equatorial analogue of the coastal Kelvin wave. In a sense, the equator acts as a wall \((v=0)\), and we have two Kelvin waves, one in each hemisphere and both propagating eastward, the northern one having the 'wall' on its right and the southern one having the 'wall' to its left. For more on equatorial waves, see Section 19-2.