AOS 311 Lecture 8 “Circulation”

**Brief description**
In this lecture we defined the circulation about a closed loop of fluid parcels and related it to the area and vorticity enclosed by the loop. We then proved that for an inviscid, homogeneous, flow the absolute circulation is conserved.

**New terminology**

**circulation**: the line integral of the tangential component of the flow about a closed loop of fluid parcels.

**baroclinic fluid**: a fluid within which the density is a function of both pressure and temperature.

**Topics covered**

1. The circulation, $C$, about a closed chain of fluid parcels is the closed line integral of the velocity tangent to the chain (loop), this is $C = \oint u \cdot dl$. Application of Stoke’s Theorem to the definition of circulation yields: $C = \iint (\nabla \times u) \cdot dA$. This form of the definition of circulation relates the circulation about a closed chain of fluid parcels to the area integral of the vorticity enclosed by that loop. From this form, we may *loosely* say that “circulation is vorticity ‘times’ area.”

2. A circulation equation, expressing the Lagrangian time rate of change of absolute circulation, $\frac{DC_a}{Dt}$, was derived for a frame of reference fixed with respect to the stars:

$$\frac{DC_a}{Dt} = D \frac{Du}{Dt} \cdot dl + \int_0^1 \left( \frac{Du}{Dt} \cdot dl + \oint u \cdot du = \oint \frac{Du}{Dt} \cdot dl + \oint \frac{|du|^2}{2} = \oint \frac{Du}{Dt} \cdot dl \right)$$

3. Substituting for Lagrangian acceleration, the three fundamental forces acting on a fluid parcel, we obtain: $\frac{DC_a}{Dt} = \oint \frac{Du}{Dt} \cdot dl = \oint \left( -\frac{\nabla p}{\rho} + g + F_r \right) \cdot dl$, where $F_r$ is frictional force. Simplifying this expression gives: $\frac{DC_a}{Dt} = \oint \frac{dp}{\rho} + \oint F_r \cdot dl$.

4. For inviscid flows, $\frac{DC_a}{Dt} = \oint \frac{dp}{\rho}$, generation or destruction of circulation is governed by the *solenoidal* term (baroclinic generation/destruction term). For inviscid, barotropic flows (for which homogenous flows are a special case), the absolute circulation is conserved. For homogeneous flows, $\frac{DC_a}{Dt} = \oint \frac{dp}{\rho} = \oint \frac{dp}{\rho} = \frac{1}{\rho} \oint dp = 0$. The last equality is valid because $dp$ is an exact differential.