AOS 311 Lecture 6 “Shallow-water systems”

**Brief description**
In this lecture, we investigated various characteristics of a shallow-water system, and allowed for generalization to homogenous, frictionless non-geostrophic flows.

**New terminology**
- **barotropic**: flows in which $u$ and $v$ are not depth varying and thus $z$-independent.

**Topics covered**

1. Observed the evolution of dye dropped into a rotating tank with a hockey puck at the bottom. We confirmed that if the topography consists of an isolated bump in an otherwise flat bottom, the fluid on the flat bottom cannot rise onto the bump, but rather goes around it. This also holds for fluid parcels at all levels above the puck, so a Taylor column was formed.

2. Shallow-water model:
   - For a homogenous, frictionless fluid ($\text{Ek} \ll 1$). The fluid does not necessarily need to be rapidly rotating, and geostrophy doesn’t necessarily apply.

   Including acceleration terms and noting that $u$ and $v$ are $z$-independent, the momentum equations become
     
     \[
     \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = \frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = \frac{1}{\rho_0} \frac{\partial p}{\partial y}.
     \]
   
   Unlike geostrophic flow, motion is not necessarily along isobars, and it can have vertical velocity.

3. We made use the continuity equation
   
   \[
   \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \right)
   \]
   
   and integrated from the bottom to top of fluid. That is:
   
   \[
   0 = \left( u_x + v_y \right) \int_{b}^{b+h} dz + \left[ w \right]_{b}^{b+h}
   \]
   
   Where $b$ is the bottom elevation above a reference level $z=0$ and $h$ is the locally, instantaneous fluid depth. We also found that the vertical velocities at levels $b$ and $b+h$ are
   
   \[
   w_{b+h} = \frac{\partial (b+h)}{\partial t} + u \frac{\partial (b+h)}{\partial x} + v \frac{\partial (b+h)}{\partial y}
   \]
   
   and
   
   \[
   w_b = u \frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y}
   \]
   
   from which we can show that the preceding integrated equation can be re-written as
   
   \[
   \frac{Dh}{Dt} = -h (u_x + v_y).
   \]
   
   This relation can be used to predict, for example, whether depth increases or decreases depending on if the vertical column in the fluid is experiencing convergence or divergence. If $(u_x + v_y) > 0$ (divergence), the depth will decrease.

4. We related pressure to depth. Using the hydrostatic equation ($\partial p / \partial z = -\rho_0 g$) and integrating from $z_{\text{top}}$ to $z$, we obtained that
   
   \[
   p(z) = p_0 - \rho_0 g z + \rho_0 g (h + b)
   \]
   
   where $p_0$ is $p_{z_{\text{top}}} = p_{\text{atm}}$ (such as the overlying atmospheric pressure) and the last term represents the dynamical pressure. It was also shown that the momentum equations can be equated to
   
   \[
   -g \frac{\partial h}{\partial x} \quad \text{or} \quad -g \frac{\partial h}{\partial y}
   \]
   
   instead of
   
   \[
   -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad \text{or} \quad -\frac{1}{\rho_0} \frac{\partial p}{\partial y},
   \]
   
   respectively.

Reading: C-R 4.3